

# Domination in Graph Theory

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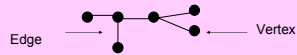
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## Introduction

### Definitions

-A **graph**  $G$  consists of nodes, formally called vertices, some of which are connected by lines, formally called edges.



- A vertex  $v$  is **adjacent** a vertex  $u$  if there is an edge from  $v$  to  $u$ . A **dominating set** of a graph  $G$  is a vertex subset with the property that every vertex of the graph is either in the dominating set or adjacent to a vertex in the dominating set. The minimum size of a dominating set is called the **domination number** and is denoted by  $\gamma(G)$ .

### Example:

Vertices =  $\{u, v, w, x, y, z\}$

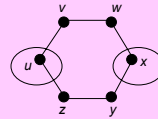
Edges =  $\{(u, v), (v, w), (w, x), (x, y), (y, z), (z, u)\}$

$u$  is adjacent to  $v$  and  $z$ .

$x$  is adjacent to  $w$  and  $y$ .

$\{u, x\}$  is a dominating set  $S$

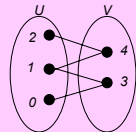
The domination number  $\gamma(G) = 2$



- A **bipartite graph** is a special graph where the set of vertices can be divided into two disjoint sets  $U$  and  $V$  such that no edge has both endpoints in the same set. An independent set in a graph is a set of pairwise nonadjacent vertices.

$U = \{0, 1, 2\}$

$V = \{3, 4\}$



$0, 1$  and  $2$  are independent vertices  
An independent set  $S = \{0, 1, 2\}$

### Motivation

The domination number of an arbitrary graph is known to be computationally challenging. An upperbound,  $k$ , on the domination number can sometimes provide an improvement for an algorithm in that sets with more than  $k-1$  vertices do not have to be considered. Thus, we queried Graffiti.pc on upperbounds for the domination number.

Even if one is willing to consider only special graphs such as bipartite graphs the computation of the domination number is still computationally challenged (i.e. there is no known algorithm that runs in polynomial time).

### Objective

The project goal for working on upper bounds for the domination number of bipartite graphs was the pursuit of a collection of upper bounds, which collectively could predict the domination number of bipartite graphs as follows. Suppose that we have a collection of (correct) inequalities of the following form

$$\gamma(G) \leq c_1, \gamma(G) \leq c_2, \dots, \gamma(G) \leq c_n$$

where  $\gamma(G)$  is the domination number of a bipartite graph  $G$  and  $c_i$  are algebraic expressions (involving graph invariants for which good algorithms are known), such that for any bipartite graph  $G$ ,

$$\gamma(G) = \text{minimum}(c_1, c_2, \dots, c_n).$$

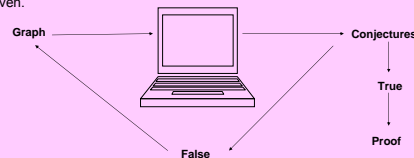
We note that such a collection would provide a method for computing the domination number of a bipartite graph. The existence of such a collection is not known; we discuss the status of our efforts in the closing comments.

### Graffiti.pc

A computer program called Graffiti.pc was the main tool used to generate conjectured upper bounds on the domination number of bipartite graphs with respect to other easily computed graph invariants. The project also included research of some mathematical literature for known upper bounds of the type described above.

Graffiti.pc is a conjecture making program, designed by Dr. E. DeLaViña (inspired by the conjecture-making program Graffiti of Dr. S. Fajtlowicz). The program receives graphs as inputs, then generates conjectures.

A **conjecture** is a statement whose truth value is unknown; it can be true or false. If the conjecture is found to be false, a graph refuting the conjecture is input to the program, otherwise the conjecture is proven.



## Conjectures

### Conjecture Summary

Of the 14 distinct conjectures that were produced by Graffiti.pc for this project, 3 were true but previously known. For the remaining 11 false conjectures a "smallest" counterexample was found, which means that it was proven to be a counterexample with the fewest possible vertices and the fewest possible edges. A simple such proof is demonstrated in the next section.

**Conjecture 1.0:** If  $G$  is a simple connected bipartite graph, then the domination number of the graph (denoted  $\gamma$ ) is the maximum degree of the graph (denoted  $\Delta$ ).

**Conjecture 2.0:** If  $G$  is a simple connected bipartite graph, then the domination number of the graph (denoted  $\gamma$ ) is the minimum degree of the graph (denoted  $\delta$ ).

**Conjecture 3.0:** If  $G$  is a simple connected bipartite graph on  $n \geq 2$  vertices, then the domination number of the graph (denoted  $\gamma$ ) is the maximum mode of degrees of the graph.

**Conjecture 4.0:** If  $G$  is a simple connected bipartite graph on at least 2 vertices, then the domination number of the graph (denoted  $\gamma$ ) equals the number of distinct minimodes of the degree of the graph.

**Conjecture 5.0:** If  $G$  is a simple connected bipartite graph on at least two vertices, then the domination number of the graph (denoted  $\gamma$ ) equals the second largest degree in the sequence of the graph (denoted  $\alpha$ ).

**Conjecture 6.0:** If  $G$  is a simple connected bipartite graph, then the domination number of the graph (denoted  $\gamma$ )  $\leq$  second largest in the sequence of the degrees of the graph.

**Conjecture 6.1 :** If  $G$  is a connected bipartite graph and  $n \geq 2$ , then  $\gamma \leq |N(A)|$ , where  $A$  is the set of vertices of minimum degree.

**Conjecture 7.0:** If  $G$  is a simple connected bipartite graph, then  $\gamma \leq$  Independence number of the graph.

*Note:* This upperbound is known to be true for all simple connected graphs on at least 3 vertices.

**Conjecture 7.1:** If  $G$  is a connected bipartite graph and  $n \geq 2$  vertices, then  $\gamma \leq |N(A) \cup U|$ , where  $A$  is the set of vertices of minimum degree.

**Conjecture 8.0:** If  $G$  is a simple connected bipartite graph, then  $\gamma =$  ceiling of (a third of number of vertices of graph).

**Conjecture 9.0:** If  $G$  is a simple connected bipartite graph, then  $\gamma \leq$  ceiling of (a third of number of vertices of graph).

**Conjecture 9.1:** If  $G$  is a simple connected bipartite graph, then  $\gamma \leq 1 + \text{min degree of complement graph}$ .

*Note:* This upperbound is known to be true for all simple connected graphs on at least 3 vertices.

**Conjecture 10.0:** If  $G$  is a simple connected bipartite graph, then  $\gamma \leq$  ceiling of  $(1 + (\text{a third of average degree of complement graph}))$ .

**Conjecture 10.1:** If  $G$  is a simple connected bipartite graph, then the domination number  $\leq 1 + \text{minimum degree of complement graph}$ .

**Conjecture 10.2:** If  $G$  is a simple connected bipartite graph, then the domination number of graph  $\leq$  half of number vertices of graph.

*Note:* This upperbound is known to be true for all simple connected graphs on at least 3 vertices.

## Applications

### Sample Proof of a Smallest Counterexample

**Conjecture 2.0:** If  $G$  is a simple connected bipartite graph, then the domination number of the graph (denoted  $\gamma$ ) is the minimum degree of the graph (denoted  $\delta$ )

The graph in Figure 1 is a counterexample to Conjecture 2.0. The proposition below demonstrates that the graph in figure 1 is a smallest counterexample to Conjecture 2.0.

**Proposition 2.0:** If  $G$  is a simple connected bipartite graph on  $n \leq 3$  vertices, then  $\gamma = \delta$ .



Figure 1



Figure 2



Figure 3

### Proof

By way of contradiction assume that  $G$  is a simple connected bipartite graph on  $n \leq 3$  vertices and  $\gamma \neq \delta$ . First, in the case that  $n = 2$ , the graph must be the path on 2 vertices and this clearly yields a contradiction to  $\gamma \neq \delta$  since  $\gamma = 1$  and  $\delta = 1$  (see Figure 2). So we assume that  $n = 3$ . Since the graph is connected and bipartite, it must be the path on 3 vertices (see figure 3). But this also clearly yields a contradiction to  $\gamma \neq \delta$  since  $\gamma = 1$  and  $\delta = 1$ . Hence, we have proved Proposition 2.0.

### Proof of True Conjecture

**Conjecture 7.0:** If  $G$  is a simple connected graph, then  $\gamma \leq$  Independence number of the graph (denoted  $\alpha$ ).

### Proof

Assume that  $G$  is a simple connected graph. Let  $A$  be a maximum independent set of  $G$ . Then there are  $\alpha$  vertices in  $A$ . Since  $A$  is a maximum independent set, every vertex not in  $A$  must be adjacent to a vertex in  $A$ . Thus,  $A$  is a dominating set and a smallest dominating set will have no more than  $\alpha$  vertices.

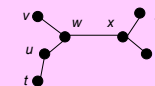
### Domination

Domination can be a useful tool for determining certain business networks and making decisions. Business would benefit from the use of the concept of domination to strategically plan the location of their stores in order to reach the maximum amount of areas with minimal stores locations.

Suppose a company wants to establish new grocery stores in a heavily populated region. The company has purchased land in the center of each neighborhood. However, the company would like to minimize the number of stores that it must build and yet still be accessible (within 10 miles of a store) to as many people in the region as possible. Let vertices of the graph be the 7 land locations purchased by the company. Two vertices will be joined by an edge if the two locations are no more than 5 miles apart.

What is the minimum number of stores that the company must build so that its stores are accessible to every one in the seven neighborhoods?

Let  $S$  represent the set of seven locations.  $S = \{t, u, v, w, x, y, z\}$



$w$  is no more than 5 miles from  $u, v$  and  $x$

$x$  is no more than 5 miles from  $w, y$  and  $z$

$t$  is no more than 5 miles from  $u$

Since  $w$  and  $x$  are accessible to a maximum number of neighborhoods, the company would benefit from putting stores at these locations. Because location  $t$  is accessible to itself and  $u$ , then a store can be located at either location. A dominating set is  $T = \{w, x, t\}$  and the domination number  $\gamma(G) = 3$ . Therefore the company would need a minimum of 3 stores to cover all seven neighborhoods.

### Closing Comments

Currently, I have found a counterexample to Conjecture 10.0 and I am in the process of proving that it is a smallest counterexample. This counterexample will be provided to Graffiti.pc and work will continue with the next list.

### References

Haynes, Teresa W., Stephen T. Hedetniemi, *Fundamentals of Domination in Graphs*, New York: Marcel Dekker, Inc., 1998.

West, Douglas B. *Introduction to Graph Theory*. Upper Saddle River: Prentice-Hall, Inc., 1996.